S1527 - Tutorials

Tutorial-1:

Exe: \$1000 $t=0$
\$200 at $t=1$
300 2
600 3
$r=5\%$ NPV

Ans:

$$NPV = -1000 + \frac{200}{1.05} + \frac{300}{(1.05)^2} + \frac{600}{(1.05)^3}$$

$$< 0$$

Exe: \$100/year 5 years
$r=4\%$ cont

Ans:

$$PV = \int_0^T f(t)e^{-rt}\,dt$$

$$= \int_0^5 100e^{-0.04t}\,dt$$

$$= 453.25$$

Exe: perpetual $t \in [0,\infty)$ $f(t)=c$ → cashflow
$r(t)$ starts at 0 and increases linearly

$r(t) = \gamma t$ $\gamma > 0$ is constant

$D_{cc}(t)$ $P(0) = \frac{P(t)}{D_{cc}(t)}$

what is general expression of $D_{cc}(t)$ for $r(t) = \gamma t$

what is PV of asset (in terms of c, γ)?

Ans: $P(0) = \frac{P(t)}{D_{cc}(t)}$ $\frac{dP}{dt} = r(t)P$

$$\frac{dP}{dt} = \gamma t P(t)$$

$$\int \frac{dP}{P} = \int \gamma t\,dt$$

$$\Rightarrow \ln\left(\frac{P(t)}{P(0)}\right) = \frac{\gamma t^2}{2}$$

$$\Rightarrow \frac{P(t)}{P(0)} = e^{\frac{\gamma t^2}{2}}$$

$$\Rightarrow P(0) = \frac{P(t)}{e^{\frac{\gamma t^2}{2}}} = \frac{P(t)}{D_{cc}(t)}$$

$$\Rightarrow D_{cc}(t) = e^{\frac{\gamma t^2}{2}}$$

$$PV(f) = \int_0^\infty f(t)e^{\frac{\gamma t^2}{2}}\,dt$$

$$= \int_0^\infty c e^{-\gamma t^2} dt$$

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \int_{-\infty}^{\infty} f_X(x)dx = 1$$

$\mu = 0, \quad \sigma = \frac{1}{\sqrt{\gamma}}$

$$\text{So, } \int_{-\infty}^{\infty} \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{x^2\gamma}{2}} dx = 1$$

$$\Rightarrow \int_0^\infty \frac{\sqrt{2\gamma}}{\sqrt{\pi}} e^{-\frac{x^2}{2}\gamma} dx = 1$$

$$\Rightarrow \int_0^\infty c\, e^{-\frac{x^2}{2}\gamma} dx = \frac{c\sqrt{\pi}}{\sqrt{2\gamma}}$$

Ex: $P(\gamma)$

$$P(\gamma) = \ln\left(\frac{\gamma+\beta}{\gamma+\alpha}\right) \qquad \beta > \alpha > 0$$

$P(\gamma) = \int_0^\infty f(t) e^{-\gamma t} dt$ what is $f(t)$?

Ans: $\mathcal{L}\{tf(t)\} = -\frac{d}{d\gamma} P(\gamma)$

$$\mathcal{L}\{f(t)\} = P(\gamma) = \int_0^\infty f(t) e^{-\gamma t} dt$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{d\gamma} P(\gamma) = \int_0^\infty t f(t) e^{-\gamma t} dt$$

$$tf(t) = \mathcal{L}^{-1}\left[-\frac{d}{d\gamma} P(\gamma)\right]$$

$$= \mathcal{L}^{-1}\left[-\left(\frac{\gamma+\alpha}{\gamma+\beta}\right)\left[\frac{(\gamma+\alpha)-(\gamma+\beta)}{(\gamma+\alpha)^2}\right]\right]$$

$$= \mathcal{L}^{-1}\left[\frac{\beta-\alpha}{(\gamma+\beta)(\gamma+\alpha)}\right] = \mathcal{L}^{-1}\left[\frac{(\gamma+\beta)-(\gamma+\alpha)}{(\gamma+\beta)(\gamma+\alpha)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{\gamma+\alpha} - \frac{1}{\gamma+\beta}\right] = e^{-\alpha t} - e^{-\beta t}$$

$$tf(t) = e^{-\alpha t} - e^{-\beta t}$$

$$f(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{t}$$

Exe: $r(t)$ follows

$$r(t) = \mu + (r_0 - \mu)e^{-\lambda t} \qquad \lambda > 0 \text{ speed of adjustment}$$

μ is long term mean

r_0 is initial rate $t = 0$

running avg $\bar{r}(T)$ over $[0, T]$

$$\bar{r}(T) = \frac{1}{T}\int_0^T r(t)\,dt$$

what is $\bar{r}(T)$?

find long term avg rate

$$\lim_{T \to \infty} \bar{r}(T)$$

Ans:

$$\bar{r}(T) = \frac{1}{T}\int_0^T \mu + (r_0 - \mu)e^{-\lambda t}\,dt$$

$$= \frac{1}{T}\left[\mu t + \frac{(r_0 - \mu)e^{-\lambda t}}{-\lambda}\right]_0^T$$

$$= \frac{1}{T}\left[\mu T + \frac{(r_0 - \mu)}{\lambda} - \frac{(r_0 - \mu)}{\lambda}e^{-\lambda T}\right]$$

$$= \mu + \frac{(r_0 - \mu)}{\lambda T}\left[1 - e^{-\lambda T}\right]$$

$\lim_{T \to \infty} \bar{r}(T) = \mu$ as $T \to \infty$ $\quad \frac{1}{T} \to 0, \quad 1 - e^{-\lambda T} \to 1$

Tutorial-2:

Exe: 3 period bin model $\Omega = \{UUU, UUD, \dots DDD\}$
$\mathcal{F}_n$ → Natural filtration generated by price process upto n

$\omega = UDU$
smallest $\mathcal{F}_2$ event contain

Ans:
$\{UDU, UDD\} \subseteq \mathcal{F}_2$

Exe: S_n → stock at n time

$$M = \{M_n\}_{n \geq 0} \quad M_n = \max(S_0, S_1, \dots S_n)$$

is M adapted with natural filteration

$$\mathcal{F}_n = \sigma(S_0, S_1 \dots S_n)$$

Ans: at every instant, value only depends on S_n
⇒ Adapted

Exe: $W = \{W_t\}_{t \geq 0}$ wiener process $W_0 = 0$ calculate $\text{cov}(W_p, W_q)$ for arbitrary points p and q s.t $0 < p < q$

Ans:

$$\text{cov}(W_p, W_q) = \mathbb{E}[(W_p - 0)(W_q - 0)]$$
$$= \mathbb{E}[W_p (W_q + W_p - W_p)]$$
$$= \mathbb{E}[W_p \cdot (Z) + W_p^2]$$

(Z independent, $Z \sim N(0, q-p)$)

$$= \mathbb{E}[W_p \cdot Z] + \mathbb{E}[W_p^2]$$
$$= \mathbb{E}[W_p^2]$$
$$= \text{var}(W_p)$$
$$= p$$

Exe: X_t → stochastic $X_t = e^{t/2} \cos(W_t)$
W_t → standard brownian motion
determine drift coeff of X_t

Ans:

$$dx = a(x,t)dt + b(x,t)dz$$

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2\right) dt + \frac{\partial G}{\partial x} b\, dz$$

$$X_t = e^{t/2} \cos(W_t)$$

$a(x,t) = 0$
$b(x,t) = 1$

$dX = dW \quad G(X) = e^{t/2} \cos(X)$
$F(W,t) = e^{t/2} \cos(W)$

$$\frac{\partial F}{\partial t} = \frac{1}{2} e^{t/2} \cos(w)$$

$$\frac{\partial F}{\partial w} = -e^{t/2} \sin(w)$$

$$\frac{\partial^2 F}{\partial w^2} = -e^{t/2} \cos(w)$$

$$\mu_F = \frac{1}{2} X_t - \frac{1}{2} X_t = 0$$

Exe: $\omega = HTTT$ $Y = \mathbb{E}[X|\mathcal{F}_2]$ $F_2 =$ inf hU first 2 tosses

Ans:

$$\mathcal{F}_2 = \sigma(\{HTXX\}, \{HHXX\}, \{THXX\}, \{TTXX\})$$

(4 options)

$$HT\begin{pmatrix} HH \\ HT \\ TH \\ TT \end{pmatrix} \begin{matrix} \to 1/4 \cdot 3 \\ \to 1/4 \cdot 2 \\ \to 1/4 \cdot 2 \\ \to 1/4 \cdot 1 \end{matrix}$$

$$\Rightarrow \frac{1}{4}[8] = 2 = \mathbb{E}[X|\mathcal{F}_2]$$

$$X|F_2 = 1 + \underbrace{\mathbb{I}_{3rd\ Head}}_{\sim Ber(Y_2)} + \underbrace{\mathbb{I}_{4th\ Head}}_{\sim Ber(Y_2)}$$

$$\Rightarrow \mathbb{E}[X|F_2] = 1 + \frac{1}{2} + \frac{1}{2}$$

$$= 2$$

Exe: $\int_0^T (dW_t)^2$

Ans:

$$dW_t = \varepsilon_t \sqrt{dt}$$

$$\Rightarrow (dW_t)^2 = \varepsilon_t^2 \, dt \to dt$$

$$\Rightarrow \int_0^T dt = T$$

Exe: S_t $dS_t = \underbrace{\tilde{\mu}}_{\text{drift}} S_t dt + \underbrace{\sigma}_{\text{volatility}} S_t dW_t$

$\mathbb{E}[S_t^2]$

Ans:

$$S_t = S_0 e^{(\tilde{\mu} - \frac{1}{2}\sigma^2)t + (\sigma W_t)}$$

$$S_t^2 = S_0 e^{(2\tilde{\mu} - \sigma^2)t + (2\sigma W_t)}$$

$$\mathbb{E}[S_t^2] = S_0 e^{(2\tilde{\mu} - \sigma^2)t} \, \mathbb{E}[e^{2\sigma W_t}]$$

$W_t \sim N(0, t)$

So, $M_{W_t}(z) = e^{t/2z^2} = \mathbb{E}[e^{zW_t}]$

MGF of $N(\mu, \sigma^2)$

$M_{W_t}(2\sigma) = e^{(t/2)(2\sigma)^2} = e^{2\sigma^2 t}$

So, $\mathbb{E}[S_t^2] = S_0 e^{(2\tilde{\mu}-\sigma^2)t} e^{2\sigma^2 t}$

Exe: W_t: $\int_0^T W_t dW_t$

Rq:

$W_t \sim N(0,t)$ $\int_0^T W_t dW_t$

$X_t = W_t^2$

$F(X,t) = X^2$

$\mu_F = \frac{1}{2}(2) = 1$

$\sigma_F = 1(2X) = 2W_t$

So, $dX_t = dt + 2W_t dW_t$

$\Rightarrow \int_0^T dX_t = \int_0^T dt + 2W_t dW_t$

$\Rightarrow X_T - X_0 = T + 2\left(\int_0^T W_t dW_t\right)$

$\Rightarrow \frac{W_T^2 - T}{2} = \int_0^T W_t dW_t$

Tutorial-3:

Exe: $\{X_t\}$ $dX_t = -2X_t dt + 4dW_t$

$\text{var}(X_t | X_0)$

Ans:

$$e^{2t} dX_t = -2X_t e^{2t} dt + 4e^{2t} dW_t$$

$$f(X,t) = e^{2t} X$$

$$dX = \underbrace{-2X}_{\mu(X,t) = -2X} dt + \underbrace{4}_{\sigma(X,t) = 4} dW_t$$

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \left(\frac{\partial f}{\partial x}\right) dB_t$$

$$= \left(2e^{2t}X + (-2X)e^{2t} + 0\right) dt + 4e^{2t} dW_t$$

$$\Rightarrow d(e^{2t} X_t) = 4e^{2t} dW_t$$

$$\int_0^t d(e^{2t} X_t) = e^{2t} X_t - X_0 = \int_0^t 4e^{2s} dW_s \sim N\left(0, \int_0^t 16 e^{4s} ds\right)$$

$$\Rightarrow e^{2t} X_t - X_0 \sim N\left(0, \frac{16}{4}[e^{4t} - 1]\right)$$

$$\Rightarrow e^{2t} X_t - X_0 \sim N(0, 4[e^{4t} - 1])$$

$$\Rightarrow e^{2t} X_t \sim N(X_0, 4(e^{4t} - 1))$$

given X_0

$$\Rightarrow X_t \sim e^{-2t} N(X_0, 4(e^{4t} - 1))$$

$$= N(e^{-2t} X_0, 4[1 - e^{-4t}])$$

$$\Rightarrow \text{var}(X_t | X_0) = 4[1 - e^{-4t}]$$

Exe: $\int_0^T (dW_t)^2$

Ans: now $dW_t = \varepsilon_t \sqrt{dt}$ $\quad \varepsilon_t \sim N(0,1)$

$$\Rightarrow (dW_t)^2 = \varepsilon_t^2 dt \to dt \text{ (conv in probability)}$$

$$\Rightarrow \int_0^T (dW_t)^2 = \int_0^T \varepsilon_t^2 dt = \int_0^T dt = T$$

$$\varepsilon_t^2 \sim \text{gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbb{E}[\varepsilon_t^2 dt] = dt$$

$$P(|\varepsilon_t^2 dt - dt| > \varepsilon) \leq \frac{\text{var}(\varepsilon_t^2 dt)}{\varepsilon^2} \to 0 \quad dt \to 0$$

$$\varepsilon_t^2 dt \xrightarrow{P} dt \Rightarrow \int_0^T (dW_t)^2 = T$$

ex: $dr_t = \lambda(\mu - r_t)\,dt + \sigma dW_t \quad \lambda, \mu, \sigma > 0$

$0 \le p \le q$

$\mathrm{Cov}(r_p, r_q)$

Ans:

$$r_t = r_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dW_s$$

$$r_p = r_0 e^{-\lambda p} + \mu(1 - e^{-\lambda p}) + \sigma \int_0^p e^{-\lambda(p-s)} dW_s$$

same for q $\qquad \sim N\left(0, \int_0^t e^{-2\lambda(t-s)} dt\right)$

$$\Rightarrow \mathbb{E}\left[\left(r_p - \mathbb{E}[r_p]\right)\left(r_q - \mathbb{E}[r_q]\right)\right]$$

$$= \mathbb{E}\left[\left(\sigma \int_0^p e^{-\lambda(p-s)} dW_s\right)\left(\sigma \int_0^q e^{-\lambda(q-s)} dW_s\right)\right]$$

$$= \sigma^2 \mathbb{E}\left[\int_0^p e^{-\lambda(p-s)} dW_s \int_0^p e^{-\lambda(q-s)} dW_s\right]$$

$$= \sigma^2 e^{\lambda[q-p]} \mathbb{E}\left[\left(\int_0^p e^{\lambda s} dW_s\right)^2\right]$$

$$= \sigma^2 e^{\lambda[q-p]} \int_0^p e^{2\lambda s} ds$$

$$= \sigma^2 e^{\lambda(q-p)} \left[\frac{e^{2\lambda p} - 1}{2\lambda}\right]$$

ex: $Y_t = \int_0^t r_u du$ find $\mathrm{var}(Y_t)$

Ans:

$$r_u = r_0 e^{-\lambda u} + \mu(1 - e^{-\lambda u}) + \sigma \int_0^u e^{-\lambda(u-s)} dW_s$$

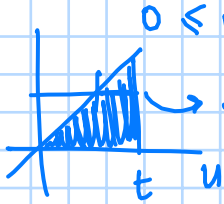
$$\int_0^t r_u du = \int_0^t r_0 e^{-\lambda u} du + \int_0^t \mu(1 - e^{-\lambda u}) du + \sigma \int_0^t \int_0^u e^{-\lambda(u-s)} dW_s\, du$$

$$= \frac{r_0}{\lambda}\left[1 - e^{-\lambda t}\right] + \mu\left[t - \left(\frac{1 - e^{-\lambda t}}{\lambda}\right)\right]$$

$$+ \text{integral}$$

$$\int_0^t \left(\int_0^u e^{-\lambda(u-s)} dW_s\right) du$$

$0 \le s \le u$, $0 \le u \le t$



$$= \sigma \int_0^t \left(\int_s^t e^{-\lambda(u-s)} du\right) dW_s$$

$$= \sigma \int_0^t e^{s\lambda} \left(\int_s^t e^{-\lambda u} du\right) dW_s$$

$$= \sigma \int_0^t e^{s\lambda} \left[\frac{e^{-\lambda y}}{-\lambda} \right]_s^t dW_s$$

$$= \frac{\sigma}{\lambda} \int_0^t e^{s\lambda} [e^{-\lambda s} - e^{-\lambda t}] dW_s$$

$$= \frac{\sigma}{\lambda} \int_0^t [1 - e^{-\lambda(t-s)}] dW_s$$

$$\text{var}\left(\int_0^t r_s ds \right) = \text{var}\left(\frac{\sigma}{\lambda} \int_0^t [1 - e^{-\lambda(t-s)}] dW_s \right)$$

$$= \frac{\sigma^2}{\lambda^2} \int_0^t \underbrace{(1 - e^{-\lambda(t-s)})^2}_{1 + e^{-2\lambda(t-s)} - 2e^{-\lambda(t-s)}} ds$$

Ex: stochastic verhulst equation

$$dX_t = (\alpha X_t - X_t^2)\, dt + \sigma X_t dW_t$$

α, σ constants

Ans:

$$dX = \mu X dt + \sigma X dW_t$$

$$X = f(t, W)$$

$$\mu f = f_t + \frac{1}{2} f_{WW}$$

$$\sigma f = f_W$$

$$\sigma dW = \frac{df}{f} \Rightarrow f = \exp(\sigma W + g(t))$$

$$g' f + \frac{\sigma^2}{2} f = \mu f$$

$$g' = \mu - \frac{\sigma^2}{2}$$

$$g = (\mu - \frac{\sigma^2}{2}) t + g_0 \in \mathbb{R}$$

$$X_t = \exp\left\{ (\mu - \frac{\sigma^2}{2}) t + (\sigma W_t) + g_0 \right\}$$

at $t = 0$ X_0 given

$$X_t = X_0 \exp\left\{ (\mu - \frac{\sigma^2}{2}) t + \sigma W_t \right\}$$

now $Y_t = \frac{1}{X_t}$

then

$$dY_t = \left((\alpha X_t - X_t^2)\left(\frac{-1}{X_t^2}\right) + (\sigma^2 X_t^2)\left(\frac{2}{X_t^3}\right) \right) dt + (\sigma X_t)\left(\frac{-1}{X_t^2}\right) dW_t$$

$$= \left(-\frac{\alpha}{x_t} + 1 + \frac{\sigma^2}{x_t} \right) dt - \frac{\sigma}{x_t} dW_t$$

$$dY_t = \left(1 + \sigma^2 Y_t - \alpha Y_t \right) dt - \sigma Y_t dW_t$$

Let $Y_t = f(t, W_t)$

$Y = f(t, W)$

$1 + \sigma^2 f - \alpha f = f_t + \frac{1}{2} f_{WW}$

$-\sigma f = f_W$

$\Rightarrow f = \exp(-\sigma W + g(t))$

$1 + \sigma^2 f - \alpha f = g' f + \frac{\sigma^2}{2} f$

$1 + \frac{\sigma^2}{2} f - \alpha f = g' f$

$g' = \frac{1}{f} + \frac{\sigma^2}{2} - \alpha$

$g' = \exp(\sigma W - g(t)) + \frac{\sigma^2}{2} - \alpha$

$dx = c_1 e^{(-x)} + c_2$

$\frac{dx}{c_1 e^{-x} + c_2} = 1$

$\int \frac{e^x dx}{c_1 + c_2 e^x} = t$

$e^x = u \quad \int \frac{du}{c_1 + c_2 u} = t + c$

$\ln |c_1 + c_2 u| = t + c$

$c_1 + c_2 e^x = ce^t$

$e^x = \frac{ce^t - c_1}{c_2}$

$e^{g(t)} = \frac{ce^t - c_1}{c_2}$ $\quad c_1, c_2$ above

$Y_t = e^{-\sigma W} e^{g(t)}$

$\frac{1}{X_t} = e^{-\sigma W} \left(\frac{ce^t - c_1}{c_2} \right)$

$X_t = \frac{c_2}{e^{-\sigma W}(ce^t - c_1)}$

$X_t = \frac{e^{\sigma W} c_2}{ce^t - c_1}$

Exe: CIR model

$$dr_t = \lambda(\mu - r_t)dt + \sigma\sqrt{r_t}\,dW_t$$

$\lambda, \mu, \sigma > 0 \quad \mathbb{E}[r_t]$

Aw: $dr_t = \lambda(\mu - r_t)dt + \sigma\sqrt{r_t}\,dW_t$

$e^{\lambda t} r_t = f(t, r_t)$

$\Rightarrow df = \left(\lambda e^{\lambda t} r_t + \lambda(\mu - r_t)e^{\lambda t}\right)dt$

$+\left(\sigma\sqrt{r_t}\, e^{\lambda t}\right)dW_t$

$\Rightarrow d(e^{\lambda t} r_t) = \lambda(\mu)e^{\lambda t}dt + \left(\sigma\sqrt{r_t}\, e^{\lambda t}\right)dW_t$

$\Rightarrow e^{\lambda T} r_T - r_0 = \mu(e^{\lambda T} - 1) + \int_0^T \sigma\sqrt{r_t}\, e^{\lambda t}\, dW_t$

$\mathbb{E} = 0$

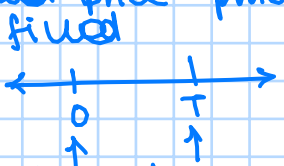
$\Rightarrow \mathbb{E}[r_T] = r_0 e^{-\lambda T} + \mu(1 - e^{-\lambda T})$

Tutorial-4:

Ex: S(0) maturity T storage cost f(t)
forward price

Ans:

forward price - price of quantity at a fixed state



if buy at s0, I would have to store it

$S_t e^{r(T-t)}$ = price of future
↑ time value

price of future ← should also count storage cost

$S_t e^{r(T-t)} + \int_t^T f(t)dt$ = price of future

as if $f(t) < S_t e^{r(T-t)} + \int_t^T f(t) e^{rt} dt$

$\pi_1 = (1, -1, 1)$
- 1: sell commodity (cost S_t)
- −1: buy forward
- 1: cost of storage saved

terminal $V_1(\pi)(0) = 0$

$$V_1(\pi)(T) = (S_t) e^{r(T-t)} + \underbrace{S_t - f_t}_{\text{forward profit we got}} - \underbrace{S_T}_{\text{sold commodity}} + \int_t^T f(t) e^{rt} dt$$

$$= -f_t + S_t e^{r(T-t)} + \int_t^T f(t) e^{rt} dt$$

$> f_t$

> 0

now if $f_t > S_t e^{r(T-t)} + \int_t^T f(t) e^{rt} dt$

$\pi_1 = (-1, 1, +)$
- −1: borrow S_t to buy it
- 1: buy stock S_t
- +: sell forward

$V_1(\pi_1)(0) = 0$

$$V_1(\pi_1)(T) = -S_t e^{r(T-t)} + S_t - \int_t^T f(t) e^{rt} dt + \underbrace{f_t - S_T}_{\text{profit/loss from forward}}$$

cost of storage

$$f_T > S_t e^{r(T-t)} + \int_t^T f(t) e^{rt}\,dt$$

$$\Rightarrow V_1(\Pi_1)(T) > 0$$

$$\text{storage cost} = \int_t^T f(s) e^{r(T-s)}\,ds \longrightarrow$$

$$S(0)e^{rT} + \int_0^T f(t) e^{r(T-t)}\,dt = F(0,T)$$

if we buy a forward, it is like



buying everything at $T=0$



apply time value

$$S(0)e^{rT} + \int_0^T f(t) e^{r(T-t)}\,dt = f(0,T)$$

$S(0)e^{rT}$: time value

$\int_0^T f(t) e^{r(T-t)}\,dt$: time value

ex: $t \in \{0,1\}$ $t=1$ up down

$B_0 = 100$ $\quad S_0 = 100$

$B_1 = 110$ $\quad S_1 = \begin{cases} 132 & \text{; up} \\ 88 & \text{; down} \end{cases}$

ex: Derivative D: \$22 if ↑, 0 if ↓

now let $\Pi_1 = (\Theta, \Phi, \eta)$ (Bond, stock, derivative)

$$100 \begin{cases} 132 \to \text{payoff } X_T(\omega_1) \\ 88 \to \text{payoff } X_T(\omega_2) \end{cases}$$

$\Pi_2 = (\psi, \theta)$ $\quad V(\Pi_2)(T) = X_T$

↑ Replicative portfolio

$$V(\Pi_2)(0) = \psi(100) + \theta(100) = c$$

and $V(\Pi_2)(T) = 110\psi + 132\theta = 22$ if up

$110\psi + 88\theta = 0$ if down

then value of portfolio same as derivative

$\Rightarrow$ C = price of derivative from law of one price

$$44\Theta = 22 \Rightarrow \Theta = 0.5$$

$$110\Psi + 44 = 0$$

$$\Psi = -\frac{44}{110}$$

$$= -\frac{4}{10}$$

$$\Psi = -0.4$$

So, $C = 100(-0.4) + 100(0.5)$

$= 100(0.1) = 10$

Tutorial-5:

Exe: K, $[0,T]$, $C_e(t)$ pays dividend D at $t_1 \in (t,T)$

$$\max(0, S(t) - De^{-r(t_1-t)} - Ke^{-r(T-t)}) \le C_e(t) \le S(t) - De^{-r(t_1-t)}$$

Ans:

$$C_e(t) > S(t) - De^{-r(t_1-t)}$$

take $\pi_1 = (1, 1, +)$ ← call derivative
- 1: Buy stock at S_0
- 1: invest borrow $C_e(t) - S(t)$
- +: sell derivative

$$V(\pi_1)(t_1^-) = (C_e(t) - S(t))e^{r(t_1-t)} + S_{t_1}^- - C_{t_1}^-$$

$$S(t_1^+) = S(t_1^-) - D$$

$$\Rightarrow V(\pi_1)(t_1^-) = (C_e(t) - S(t))e^{r(t_1-t)} + S_{t_1}^+ + D - C_{t_1}^-$$

$$V(\pi_1)(t_1^+) = (C_e(t) - S(t))e^{r(t_1-t)} + S_{t_1} + D - C_{t_1}$$

(invest D in risk free asset)

$\pi_2 = (1, 1, +)$

$$V(\pi_2)(T) = [C_e(t) - S(t)]e^{r(T-t)} + De^{r(T-t_1)} + S_T + \min(0, K - S_T)$$

$$= e^{r(T-t)}\left[C_e(t) - S(t) - De^{-r(t_1-t)}\right] + S_T + \min(0, K - S_T)$$

($e^{r(T-t)}$ and $e^{r(T-t_1-T+t)}$; $De^{-r(t_1-t)} > 0$; $\min(0, K - S_T) > 0$)

$$\Rightarrow V(\pi_2)(T) > 0$$

$$C(t) < \max(0, S_t - De^{-r(t_1-t)} - Ke^{-r(T-t)})$$

WLOG $C(t) < S_t - De^{-r(t_1-t)} - Ke^{-r(T-t)}$

$$\pi_1 = (1, +, 1)$$

↑ invest $S_t - C(t)$

$$V(\pi_1)(t_1^-) = [S_t - C_t]e^{r(t_1-t)} - S_{t_1}^- + C_t$$

$$S_{t_1}^+ = S_{t_1}^- - D$$

$$= [S_t - C_t]e^{r(t_1-t)} - S_{t_1}^+ - D + C_t$$

$$\pi_2 = ((1,1), +, 1)$$

$$V(\pi_2)(T) = [S_t - C_t - De^{-r(t_1-t)}]e^{r(T-t)} - S_T + \max(0, S_T - K)$$

($[S_t - C_t - De^{-r(t_1-t)}]e^{r(T-t)} > K$)

$$> K - S_T + \max(0, S_T - K) > 0$$

Exe: Show no such that

there ∃ arbitrage opportunity $C_e(t) - P_e(t) = S(t) - Ke^{-r(T-t)}$

Ans:

If $C_e(t) - P_e(t) = S(t) - Ke^{-r(T-t)} > S(t) - Ke^{-r(T-t)} - De^{-r(t_1-t)}$

$C_e(t) - P_e(t) > S(t) - Ke^{-r(T-t)} - De^{-r(t_1-t)}$

$\pi_1 = (1, 1, (-1, 1))$

↑ buy $S(t)$ ↑ buy put ↑ sell call

cost/borrow $-P_e(t) - S(t) + C_e(t)$

$V(\pi_1)(t_1^-) = -[P_e(t) + S(t) - C_e(t)]e^{r(t_1-t)}$

$+ S_{t_1} + D + P_{t_1} - C_{t_1}$

↑ put this in bank

$V(\pi_2)(T) = [P_e(t) - S(t) + C_e(t) + De^{-r(t_1-t)}]e^{r(T-t)}$

(>0) $+ \max(0, K - S_T) - \min(0, K - S_T) + S_T$

(>0) (>0)

⇒ this is an arbitrage

So, $C_e(t) - P_e(t) = S_t - De^{-r(t_1-t)} - Ke^{-r(T-t)}$